Analyze existence, uniqueness and controllability of impulsive fractional functional differential equations

K. Muthuselvan[1] and B. Sundara Vadivoo[2]

[1,2]Department of Mathematics, Alagappa University, Karaikudi-630 003
[2]Corresponding author
E-mail: sundaravadivoob@alagappauniversity.ac.in

Abstract

This manuscript demonstrated the impulsive fractional functional differential equation and established the controllability criterion. By employing Laplace transformation and Mittag-Leffler function, the solution representation was derived and subsequently, one can construct the suitable control function and analyzed the controllability criteria for the given dynamical system. The existence results acquired with some assumptions with Schauder Fixed Point theorem and uniqueness results were attained by Banach Contraction Principle. Eventually, two numerical examples were provided with MATLAB graphical representation for the efficacy of results.

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1 Introduction

The fractional differential equation was one of the renowned areas in Mathematics. It provided a magnificent contrivance for the description of memory and hereditary properties of various materials and processes. Fractional calculus was widely applicable in non-integer orders. Fractional derivatives can be employed in various fields like Fluid Flows [18], Engineering [18], Viscoelasticity [3], Control Theory [23], and Nano-Technology [14]. Recently, fractional differential equations can be achieved in the field of modelling and the medical field. It was envisioned the finest application in a non-linear dynamical system of earthquakes, weather reporting, and economics. In effect of fractional differential equations, one can examine the surrogate model to integer differential equations. In most challenging task for a mathematician was describing the model by differential equations contains fractional order. The progression of fractional calculus was much more complicated way than in the classical integer-order. Many physical processes emerged to evince fractional-order behavior that may vary with time (or) space. The fractional calculus had been dealing with the differentiation and integration to any fractional order. Caputo fractional derivatives were one of the finest advantages allowed initial and boundary conditions. Also, a constant derivative of Caputo was zero but in Riemann-Liouville derivative constant, need not be zero.

Impulsive differential equations have played a cardinal task in modelling sight, especially in describing the populations growth. The decay rate subject was to abrupt changes as well as other fields such as harvesting, diseases, and so forth. In many advancement processes of a dynamical system, there occurs a sudden change such as disturbance, reap (or) natural disaster. The discontinuous jump in this process can be generally called as impulses. It can be classified into two sections; one is instantaneous and another one is non-instantaneous impulses. In the study of instantaneous

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impulses, the duration of the changes was relatively short compared to the overall duration of the whole process. In case of non-instantaneous impulses, the action of impulses starts abruptly at a fixed point and its action continuous up to some finite time interval. Examples of instantaneous impulses in real-life situations were earthquakes, Tsunami, and so on. In non-instantaneous impulses was the introduction of the drugs, anaesthesia, and pain killer medicine in the bloodstream. It takes some finite time to make changes on the action of the body. The continuous process in the locus was an example of non-instantaneous impulses. Also, impulsive fractional functional differential equations, used to model the biological system, nano particles, and so forth.

Generally, most of the dynamical systems were examine in either continuous or discrete-time intervals, many systems in the fields of physics, biology, chemistry and engineering. The information science may understand abrupt changes as certain instants during the continuous dynamical system. On the other hand, there had been a new category of a dynamical system, which is neither purely continuous-time nor purely discrete-time ones these are called impulsive control systems.

The controllability played the finest contribution in the highly excellent reaction of a dynamical system. The conceptualization of control theory was based on the mathematical description of the dynamical system. Generally, it can be actions of govern to control a dynamical system from their input based on the simultaneous observations of outputs with the assistance of some set of admissible control. Hence, the combination of fractional-order derivatives and integrals in the theory of controllability accords to the finest results than the integer-order derivatives.

In this manuscript, we concern about the concept of controllability on impulsive fractional functional differential equations. It means that the system is transferred from the initial state to desired final state by using a set of admissible control. In the literature survey, K.Balachandran.et.al. the controllability of fractional integro-differential systems in Banach spaces [6]. M.Belmekki.et.al the Existence results for fractional order semilinear Functional Differential equations with non dense Domain [8]. F. Chen.et.al, On the Solutions for Impulsive Fractional Differential Equations [10]. X. B. Shu et.al. Existence and Uniqueness of Mild Solution for Abstract Fractional Functional Differential Equations [26]. Recently, B.Sundara Vadivoo.et.al developed the controllability criteria of fractional differential dynamical systems with non-instantaneous impulses proved by [28]. From this survey, there is no manuscript considering the Existence, Uniqueness and Controllability of impulsive fractional functional differential equations with the following approach. So motivated by this fact in this paper we will define in the following impulsive fractional functional differential equations.

$$\begin{aligned} {}^{C}D^{\alpha}x(t) &= Hx(t) + Pu(t) + f(t, x_t), t \in J := [0, T] - \{t_1, t_2, ...t_m\} \\ \Delta x(t) &= x(t_i^+) - x(t_i^-) = I_i(x(t_i^-)), , i = 1, 2, 3, ...m \\ x(t) &= \varphi(t), t \in \mathcal{K} =: [-r, 0] \end{aligned} \tag{1.1}$$

where ${}^{C}D^{\alpha}$ represents the Caputo derivative of order $0 < \alpha < 1$, $H \in \mathbb{R}^{n\times n}$ and $P \in \mathbb{R}^{n\times m}$ are constant matrices, $u \in \mathbb{R}^m$ is an admissible control function and the state vector $x(t) \in PC^1[-r, T]$. Let the function $f : J\times\mathcal{K} \to \mathbb{R}^n$ is Lebesgue measurable function with respect to t on J is continuous. The impulsive function $I_i : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, i=1,2,3,...m and $\varphi : \mathcal{K} \to \mathbb{R}^n$ with sup-norm, $\|\varphi\|_\infty = \sup_{-r\leq s\leq 0} \|\varphi(s)\|$, where r is a positive constant. Let $x_t \in \mathcal{K}$ be a continuous function is defined by $x_t(s) = x(t + s)$ for $-r \leq s \leq 0$, here x_t represents history of the state from time $t - r$ up to present time t. Let $x(t_i^+) = \lim_{\varepsilon\to 0^+}(t_i + \varepsilon)$ and $x(t_i^-) = \lim_{\varepsilon\to 0^-}(t_i + \varepsilon)$ is represents the right and left limit of x(t) at $t = t_i$ and the discontinuous points are

$$0 = t_0 < t_1 < t_2 < .... < t_m < t_{m+1} = T < \infty$$

The finest contribution of the research work affirms as follows:

- Consider the Fractional Functional differential equations with impulses.
- Solution representation of our system by manipulating the Laplace transform and Mittag-Leffler function.
- Constructing the suitable control function and linear operator for given dynamical model.
- Existence results were effectuated from some assumptions and fixed point theorem.
- Executing the controllability criteria by using Banach Contraction Principle for the given impulsive fractional functional differential equation.
- The novelty of this manuscript was to find the existence, uniqueness, and controllability of a system utilizing specific assumptions. It had never been used in a prior research study, as well as to develop appropriate control functions for the given system. This assumption and control function will produce more effective outcome than earlier studies [6],[8],[10],[26] and [28], as well as provide examples of the veracity of our assumptions.
- The originality of this paper was to establish the controllability results by the way of transfer the control function into the linear operator and to obtaining these results without semi-group theory.

The contour of the manuscript is organized as follows, In section 2, we present the the basic definition of concern for proving main results. In section 3, providing the solution representation of the fractional functional system with impulses. In section 4, to initiate the existence and uniqueness results of a given dynamical system and also execute the controllability criteria. In section 5, two numerical examples were provided. In section 6, we gave the graphical representation solution of the system and Eventually, In section 7, We described the conclusion of the manuscript.

2 Preliminaries

This section presented some preliminary concepts, such as definitions and Lemmas, that will help us to prove that the controllability criterion for impulsive fractional functional differential equations.

Definition 2.1. [33] Let $J =: [a, b](-\infty < a < b < \infty)$ be a finite interval of $\mathbb{R}$. The left and right Riemann-Liouvile fractional integrals ${}_aD_t^{-\alpha}f(t)$ and ${}_tD_b^{-\alpha}f(t)$ of order $\alpha \in \mathbb{R}^+$ and $f \in \mathcal{L}_1(\mathbb{R}^+)$ are defined by

$$ {}_aD_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \ t > a, \ \alpha > 0$$

$$ {}_tD_b^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds, \ t < b, \ \alpha > 0$$

Definition 2.2. [33] The Caputo fractional derivative of order $n-1<\alpha<n$ is defined by:

$$ {}_{a}^{C}D_{t}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}f^{(n)}(s)ds,\ t>a $$

$$ {}_{t}^{C}D_{b}^{\alpha}f(t)=\frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{t}^{b}(s-t)^{n-\alpha-1}f^{(n)}(s)ds,\ t<b $$

Where f(t) have absolutely continuous derivatives order upto (n-1).

Definition 2.3. [33] A two parameter of Mittag-Leffler function is defined by the series expansion

$$ E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)} $$

for $\alpha,\beta>0$ and $\mathbb{Z}\in\mathbb{C}$, Where $\mathbb{C}$ denotes the complex plane.

Definition 2.4. [25] Let f(t) be defined for $0\leq t<\infty$ and let 's' denote on arbitrary real variable. The Laplace transform of f(t) designated by either $\mathcal{L}\{f(t)\}$ or F(s), is

$$ \mathcal{L}\{f(t)\}=F(s)=\int_{0}^{\infty}e^{-st}f(t)dt $$

for all values of 's' for which the improper integral converges. Convergence occur when the limit

$$ \lim_{R\to\infty}\int_{0}^{\infty}f(t)dt $$

exists. If limit does not exists the improper integral diverges and f(t) has no Laplace transform and f(t) has n-dimensional vector valued function.

Definition 2.5. [33] A family F in C(J,X) is called uniformly bounded if there exists a positive constant K such that $|f(t)|\leq K$ for all $t\in J$ and all $f\in F$, where $J=[a,b]$, $(-\infty<a<b<\infty)$ be a finite interval.

Definition 2.6. [33] A family F in C(J,X) is called equicontinuous, if for every $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $|f(t_1)-f(t_2)|<\varepsilon$ for all $t_1,t_2\in J$ with $|t_1-t_2|<\delta$ and all $f\in F$, where $J=[a,b]$, $(-\infty<a<b<\infty)$ be a finite interval.

Note 2.7. [33] Denote C(J,X) be the Banach space of all continuous function from J into X with the norm

$$ \|x\|_{\infty}=\sup_{t\in J}\|x(t)\|,\ x\in C(J,X) $$

where $J=[a,b]$, $(-\infty<a<b<\infty)$ be a finite interval.

Note 2.8. [30] Let $C^1([-r,T],X)$ denotes the space of left continuous derivative function from $[-r,T]$ to $\mathbb{R}^n$ with norm

$$ \|x\|_{\infty}=\sup_{t\in[-r,T]}\|x(t)\|,\ x\in C^{1}([-r,T],\mathbb{R}^{n}),\ \textit{where r is a positive constant.} $$

Definition 2.9. [5] A system is said to be controllable on [0,T] if every pair of vector $x_0, x_1 \in \mathbb{R}^n$ there exists a control $u \in L^2_m[0,T]$ such that the solution x which satisfies $x(0) = x_0$ and $x(T) = x_1$.

Lemma 2.10. [5] Let X be a real Banach space, $M \subset X$ a non-empty closed bounded convex subset and $F : M \to M$ is compact then F has a fixed point.

Lemma 2.11. [33] Let X be a Banach space and W $\subset PC(J, X)$, where $J = [a, b]$, $(-\infty < a < b < \infty)$ be finite interval. If the following conditions are satisfied:
(i) W is uniformly bounded subset of PC(J, X);
(ii) W is equicontinuous in (t_i, t_{i+1}), k = 0, 1,2, . .,m, where $t_0 = 0, t_{m+1} = T$;
(iii) $W(t) = \{\ x(t) : x \in W, t \in J - \{t_1, t_2, t_3, ...t_m\}\}$, $W(t_i^+) = \{x(t_i^+) : x \in W\}$ and $W(t_i^-) = \{x(t_i^-) : x \in W\}$ are relatively compact subset of X. Then W is a relatively compact subset of PC(J, X).

Remark 2.12. [33] A subset F in C(J,$\mathbb{R}$) is relatively compact if and only if it is uniformly bounded and equicontinuous on J, where $J = [a, b]$, $(-\infty < a < b < \infty)$ be a finite interval.

Lemma 2.13. [28] Let $\alpha \in (0, 1)$ and $\chi : J \to \mathbb{R}$ is continuous, where $J = [0, T]$. A function $x \in C(J, \mathbb{R})$ is a solution of the fractional integral equations

$$x(t) = x_0 - \frac{1}{\Gamma(q)} \int_0^h (h - s)^{q-1} \chi(s) ds + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \chi(s) ds$$

if and only if x is a solution of the following fractional Cauchy problem,

$$\begin{aligned} {}^C D^\alpha x(t) &= \chi(t), t \in J \\ x(h) &= x_h, h > 0 \end{aligned}$$

Lemma 2.14. [28] Let $\mathbb{C}$ be a complex plane for any $\alpha > 0$, $\beta > 0$ and $H \in \mathbb{C}^{n\times n}$ then

$$\mathcal{L}\{t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(Ht^\alpha)\} = \frac{s^{\alpha-\beta} k!}{(s^\alpha - H)^{k+1}}, \mathbb{R}(s) > \|H\|^{\frac{1}{\alpha}}. \tag{2.1}$$

In particular,

$$\mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(Ht^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha - H}, \mathbb{R}(s) > \|H\|^{\frac{1}{\alpha}}. \tag{2.2}$$

3 Solution representation

In this section, we discussed about the solution representation of our given system.

Lemma 3.1. Let $0 < \alpha < 1$ and $f : J \times \mathcal{K} \to \mathbb{R}^n$ be a Lebesgue measurable continuous function then the equation. (1.1) is defined without impulses as follows,

$$\begin{aligned} {}^C D^\alpha x(t) &= Hx(t) + Pu(t) + f(t, x_t), t \in J := [0, T] - \{t_1, t_2, ...t_m\} \\ x(t) &= \varphi(t), t \in [-r, 0] \end{aligned} \tag{3.1}$$

The equation.(3.1) is equivalent in the form of following fractional functional differential equation

$$x(t) = \begin{cases} \varphi(t), & t \in [-r, 0]; \\ E_\alpha(Ht^\alpha)\varphi(0) + \int_0^t (t - s)^\alpha E_{\alpha,\alpha} H(t - s)^\alpha [Pu(s) + f(s, x_s)]ds, & t \in [0, T]; \end{cases}$$

Proof. Let us consider the IVP for fractional functional differential in $[0, T]$ and employing Laplace Transformation for the equation.(3.1) then we get,

$$\begin{aligned} \mathcal{L}\{{}^C D^\alpha x(t)\} &= \mathcal{L}\{Hx(t)\} + \mathcal{L}\{Pu(t) + f(t, x_t)\} \\ s^\alpha X(s) - s^{\alpha-1}x(0) &= HX(s) + \mathcal{L}\{Pu(t) + f(t, x_t)\} \\ X(s) &= s^{\alpha-1}x(0)(s^\alpha I - H)^{-1} + (s^\alpha I - H)^{-1}\mathcal{L}\{Pu(t) + f(t, x_t)\}. \end{aligned} \tag{3.2}$$

Applying inverse Laplace transform in the above equation then we get,

$$\mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\{s^{\alpha-1}x(0)(s^\alpha I - H)^{-1} + (s^\alpha I - H)^{-1}\}\{Pu(t) + f(t, x_t)\}. \tag{3.3}$$

By using **Lemma.(2.14)** and Convolution theorem, equation.(3.3) becomes

$$x(t) = E_\alpha(Ht^\alpha)\varphi(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha\{Pu(s) + f(s, x_s)\}ds, t \in [0, T].$$

Q.E.D.

Lemma 3.2. Let $0 < \alpha < 1$ and $f : J \times \mathcal{K} \to \mathbb{R}^n$ be a Lebesgue measurable continuous function with respect to t on J. A function $x \in (PC^1([-r, T], \mathbb{R}^n)$ is a solution of the fractional functional differential equations,

$$x(t) = \begin{cases} \varphi(t), & t \in [-r, 0]; \\ E_\alpha(Ht^\alpha)\varphi(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha[Pu(s) + f(s, x_s)]ds, & t \in (0, t_1]; \\ E_\alpha(Ht^\alpha)\varphi(0) + I_1(x(t_1^-)) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha & \\ \times[Pu(s) + f(s, x_s)]ds, & t \in (t_1, t_2]; \\ . & \\ . & \\ . & \\ E_\alpha(Ht^\alpha)\varphi(0) + \sum_{i=1}^m I_i(x(t_i^-)) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha & \\ \times[Pu(s) + f(s, x_s)]ds, & t \in (t_i, t_{i+1}]; \end{cases}$$

if and only if x is a solution of the equation.(1.1).

Proof. Assume that 'x' is a solution of the impulsive fractional functional differential equation.(1.1) If $t \in (0, t_1]$ then we have

$$\begin{aligned} {}^C D^\alpha x(t) &= Hx(t) + Pu(t) + f(t, x_t), t \in J := [0, T] - \{t_1, t_2, ...t_m\} \\ x(0) &= \varphi(0) \end{aligned} \tag{3.4}$$

Integrating the above equation from 0 to t by using the **Lemma.(3.1)**, one can obtain

$$x(t) = E_\alpha(Ht^\alpha)\varphi(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha[Pu(s) + f(s, x_s)]ds$$

and

$$x(t_1^-) = E_\alpha(Ht^\alpha)\varphi(0) + \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha} H(t_1 - s)^\alpha [Pu(s) + f(s, x_s)]ds. \tag{3.5}$$

If $t \in (t_1, t_2]$ then the equation.(1.1) in the form of

$$\begin{aligned} {}^C D^\alpha x(t) &= Hx(t) + Pu(t) + f(t, x_t), t \in J := [0, T] \ - \{t_1, t_2, ...t_m\} \\ \Delta x(t) &= x(t_1^+) - x(t_1^-) = I_1(x(t_1^-)) \end{aligned} \tag{3.6}$$

By using **Lemma.(2.13)** and equation.(3.5) then we get

$$\begin{aligned} x(t) &= x(t_1^+) - \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha} H(t_1 - s)^\alpha [Pu(s) + f(s, x_s)ds \\ &+ \int_0^t (t - s)^{\alpha-1} E_{\alpha,\alpha} H(t - s)^\alpha [Pu(s) + f(s, x_s)]ds \\ &= x(t_1^-) + I_1(x(t_1^-)) - \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha} H(t_1 - s)^\alpha [Pu(s) + f(s, x_s)]ds \\ &+ \int_0^t (t - s)^{\alpha-1} E_{\alpha,\alpha} H(t - s)^\alpha [Pu(s) + f(s, x_s)]ds \\ &= E_\alpha(Ht^\alpha)\varphi(0) + \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha} H(t_1 - s)^\alpha [Pu(s) + f(s, x_s)]ds + I_1(x(t_1^-)) \\ &- \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha} H(t_1 - s)^\alpha [Pu(s) + f(s, x_s)]ds \\ &+ \int_0^t (t - s)^{\alpha-1} E_{\alpha,\alpha} H(t - s)^\alpha [Pu(s) + f(s, x_s)]ds \end{aligned}$$

$$\begin{aligned} &= E_\alpha(Ht^\alpha)\varphi(0) + I_1(x(t_1^-)) + \int_0^t (t - s)^{\alpha-1} E_{\alpha,\alpha} H(t - s)^\alpha [Pu(s) + f(s, x_s)]ds \\ x(t) &= E_\alpha(Ht^\alpha)\varphi(0) + I_1(x(t_1^-)) + \int_0^t (t - s)^{\alpha-1} E_{\alpha,\alpha} H(t - s)^\alpha [Pu(s) + f(s, x_s)]ds \end{aligned}$$

Proceeding like this, in general if for every $t \in (t_i, t_{i+1}]$, i=1,2,3,...,m then the equation.(1.1) in the form of

$$\begin{aligned} {}^C D^\alpha x(t) &= Hx(t) + Pu(t) + f(t, x_t), t \in J := [0, T] - \{t_1, t_2, ...t_m\} \\ \Delta x(t) &= x(t_i^+) - x(t_i^-) = I_i(x(t_i^-)), \ i = 1, 2, 3, ...m \end{aligned}$$

and the solution representation of above equation is

$$x(t) = E_\alpha(Ht^\alpha)\varphi(0) + \sum_{i=1}^{m} I_i(x(t_i^-)) + \int_0^t (t - s)^{\alpha-1} E_{\alpha,\alpha} H(t - s)^\alpha [Pu(s) + f(s, x_s)]ds. \tag{3.7}$$

Similarly, the converse part of the lemma can be proved. Q.E.D.

4 Main results

In this section, we explore the existence and uniqueness solutions and also prove the controllability criteria for the equation.(1.1).

4.1 Existence results

Now we define a mapping for the function $N : PC^1[-r,T] \to PC^1[-r,T]$ and satisfied all the assumptions for proving the controllability criteria for the equation.(1.1),

$$(Nx)(t) = \begin{cases} \varphi(t) & t \in [-r,0]; \\ E_\alpha(Ht^\alpha)\varphi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha [Pu(s) + f(s,x_s)]ds, & t \in (0,t_1]; \\ E_\alpha(Ht^\alpha)\varphi(0) + I_1(x(t_1^-)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha & \\ \times[Pu(s) + f(s,x_s)]ds, & t \in (t_1,t_2]; \\ . & \\ . & \\ . & \\ . & \\ E_\alpha(Ht^\alpha)\varphi(0) + \sum_{i=1}^m I_i(x(t_i^-)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha & \\ \times[Pu(s) + f(s,x_s)]ds, & t \in (t_i,t_{i+1}]; \end{cases}$$

We can introduce some following assumptions

A1. A function $f : J \times \mathcal{K} \to \mathbb{R}^n$ and there exists a constant 'm' such that $\|f(t,x)\| \leq m(t)$, for every $t \in [0,T]$, $m > 0$, $x \in J = [0,T]$.

A2. A function $f : J \times \mathcal{K} \times \to \mathbb{R}^n$ and there exists a constant 'm' such that $\|f(t,x) - f(t,y)\| \leq m(t)\|x-y\|_\infty$, for every $t \in [0,T], x,y \in \mathbb{R}^n, m > 0$.

A3. The Impulsive function $I_i : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists a constant $\lambda > 0$ such that $\|I_i(x(t_i^-)) - I_i(y(t_i^-))\| \leq \lambda\|x-y\|_\infty$ for every $x,y \in \mathbb{R}^n, \lambda > 0$ and $i = 1,2,3,...m$.

A4. The linear operator $\kappa : L^2(u,[0,T]) \to R^n$, is defined by

$$\kappa u = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(H(t-s)^\alpha) Pu(s)ds.$$

has an invertible operator κ^{-1} taking values in $L^2(u,[0,T])/ker(\kappa)$ and $t \in [0,T]$ then there exists a positive constant $\eta > 0$ such that $\|\kappa^{-1}\| \leq \eta$.

Lemma 4.1. Assume (A1)-(A4) are satisfied then the control function for the fractional functional differential equation has assess $\|u(t)\| \leq Q^\star$, for every $t \in (t_i,t_{i+1}]$ such that

$$Q^\star := \eta\|x_t\| + \|E_\alpha(Ht^\alpha)\varphi(0)\| + M^\star + \int_0^t \|(t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha\| m(s)ds. \tag{4.1}$$

Proof. Let us defined a control function for the equation.(1.1)

$$u(t) = \kappa^{-1}[x_t - E_\alpha(Ht^\alpha)\varphi(0) - \sum_{i=1}^m I_i(x(t_i^-)) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha f(s,x_s)ds. \tag{4.2}$$

we substitute $t = t_{i+1}$ in equation.(3.7) and using the assumption (A4) and equation.(4.2) then we get in the form of

$$\begin{aligned} x(t_{i+1}) &= E_\alpha(Ht_{i+1}^\alpha)\varphi(0) + \sum_{i=1}^m I_i(x(t_i^-)) + \int_0^{t_{i+1}} (t_{i+1}-s)^{\alpha-1}E_{\alpha,\alpha}H(t_{i+1}-s)^\alpha f(s,x_s)ds \\ &\quad + \kappa\kappa^{-1}[x_{t_{i+1}} - E_\alpha(Ht_{i+1}^\alpha)\varphi(0) - \sum_{i=1}^m I_i(x(t_i^-)) \\ &\quad - \int_0^{t_{i+1}} (t_{i+1}-s)^{\alpha-1}E_{\alpha,\alpha}H(t_{i+1}-s)^\alpha f(s,x_s)]ds \\ x(t_{i+1}) &= x_{t_{i+1}}. \end{aligned}$$

The control function of equation.(4.2) can be defined as follows

$$\|u(t)\| \le \eta\|x_t\| + \|E_\alpha(Ht^\alpha)\varphi(0)\| + M^\star + \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha\|m(s)ds. \tag{4.3}$$

where, $\sum_{i=1}^m \|I_i x(t_i^-)\| = M^\star$, $\|\kappa^{-1}\| = \eta$. Q.E.D.

Theorem 4.2. Assume that the conditions (A1)-(A4) are satisfied then the equation.(1.1) has at least one solution on $J = [0,T]$.

$$\|E_\alpha(Ht^\alpha)\varphi(0)\| + M^\star + \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha\|[\tilde{\kappa}Q^\star + m(s)]ds \le \rho. \tag{4.4}$$

where, $\sum_{i=1}^m \|I_i x(t_i^-)\| = M^\star$, $\|P\| = \tilde{\kappa}$, $\rho \in \mathbb{R}^+$.

Proof. **Step 1: N maps from bounded sets into itself**

To Prove a function N maps from bounded set into itself by using **Lemma.(3.2)** and according to Nx(t), for $t \in (0, t_1]$ we get the following inequality,

$$\begin{aligned} \|(Nx)(t)\| &= \|E_\alpha(Ht^\alpha)\varphi(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha[Pu(s) + f(s,x_s)ds]\| \\ &\le \|E_\alpha(Ht^\alpha)\varphi(0)\| + \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha Pu(s)ds\| \\ &\quad + \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha f(s,x_s)ds\| \\ &\le \|E_\alpha(Ht^\alpha)\varphi(0)\| + \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha\| \times [\tilde{\kappa}Q^\star + m(s)]ds \\ &\le \rho_1. \end{aligned} \tag{4.5}$$

Continuing in this manner, in general form for every $t \in (t_i, t_{i+1}]$ then we have,

$$\begin{aligned} \|(Nx)(t)\| &= \|E_\alpha(Ht^\alpha)\varphi(0) + \sum_{i=1}^m I_i(x(t_i^-)) \\ &\quad + \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha[Pu(s) + f(s,x_s)ds\| \end{aligned}$$

$$\begin{aligned}
&\leq \|E_\alpha(Ht^\alpha)\varphi(0)\| + \sum_{i=1}^{m} \|I_i(x(t_i^-))\| + \int_0^t \|(t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha Pu(s)ds\| \\
&\quad + \int_0^t \|(t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha f(s,x_s)ds\| \\
&\leq \|E_\alpha(Ht^\alpha)\varphi(0)\| + \sum_{i=1}^{m} M_i + \int_0^t \|(t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha\| \times [\tilde{\kappa}Q^\star + m(s)]ds \\
&\leq \rho_i \qquad (4.6)
\end{aligned}$$

where $\sum_{i=1}^{m} M_i = \sum_{i=1}^{m} \|I_i(x(t_i^-))\|$, $\|u(s)\| = Q^*$, $\|P\| = \tilde{\kappa}$.

From equations.(4.5) and (4.6) we have,

$$\begin{aligned}
\|(Nx)(t)\| &= max\{\|E_\alpha(Ht^\alpha)\varphi(0)\| + M_1 + \int_0^t \|(t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha\| \times [\tilde{\kappa}Q^\star + m(s)]ds, \\
&\quad ..., \|E_\alpha(Ht^\alpha)\varphi(0)\| + \sum_{i=1}^{m} M_i + \int_0^t \|(t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha\| \times [\tilde{\kappa}Q^\star + m(s)]ds\} \\
&\leq max\{\rho_1, ..., \rho_i\} \\
&= \rho \\
\Rightarrow \|(Nx)(t)\| &\leq \rho. \qquad (4.7)
\end{aligned}$$

From equation.(4.7) we can say the function N is bounded.

Step 2: N is continuous

Let $\{x^v\}$ be a sequence on $PC^1([-r,T])$ such that $\{x^v\} \to x$ (say) as $v \to \infty$. For each $t \in [-r,0]$, $\|(Nx^v)(t) - (Nx)(t)\| = \|\varphi(t) - \varphi(t)\| \equiv 0$ and then we have

$$\|(Nx^v)(t) - (Nx)(t)\| \to 0 \ as \ v \to \infty \qquad (4.8)$$

From equation.(4.8) the function N is continuous on $[-r,0]$. For every $t \in (0,t_1]$ then we have,

$$\begin{aligned}
\|(Nx^v)(t) - (Nx)(t)\| &= \|E_\alpha Ht^\alpha\varphi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha Pu(s)ds \\
&\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha f(s,x_s^v)ds - E_\alpha Ht^\alpha\varphi(0) \\
&\quad - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha Pu(s)ds \\
&\quad - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha f(s,x_s)ds\| \\
&\leq \int_0^t \|(t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha [f(s,x_s^v) - f(s,x_s)]ds\| \\
&\leq \int_0^t \|(t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha\| \times \|[f(s,x_s^v) - f(s,x_s)]\|ds \\
&\leq \int_0^t \|(t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha\| \times m(s)\|x^v - x\|_\infty \, ds.
\end{aligned}$$

Since, as $v \to \infty$, x^v is convergent to x, $t \in (0, t_1]$ then we get the function $\|(Nx^v)(t)-(Nx)(t)\| \to 0$ as $v \to \infty$ from this we can say the function N is continuous on $(0, t_1]$.
Proceeding like this, in general for every $t \in (t_i, t_{i+1}]$ then we get in the form of

$$\|(Nx^v)(t) - (Nx)(t)\| \leq \|x^v - x\|_\infty \{\sum_{i=1}^{m} \lambda_i + \int_0^t \|(t-s)^{\alpha-1} E_{\alpha,\alpha} H(t-s)^\alpha\| \times m(s)ds\}.$$

Since, as $v \to \infty$, x^v is convergent to x, $t \in (t_i, t_{i+1}]$ then we get the function $\|(Nx^v)(t) - (Nx)(t)\| \to 0$ as $v \to \infty$ from this we can say the function N is continuous on $(t_i, t_{i+1}]$, i=1,2,...,m.

Step 3: N is equicontinuous

The function N is bounded and continuous from step 1 and step 2 respectively. We want to show that the function N is equicontinuous. Now, for each $t \in [-r, 0]$, $\|(Nx^v)(t)-(Nx)(t)\| = \|\varphi(t)-\varphi(t)\| \equiv 0$ and then we have

$$\|(Nx^v)(t) - (Nx)(t)\| \to 0 \; as \; v \to \infty$$

From above equation the function N is equicontinuous on $[-r, 0]$. Let us consider the two arbitrary elements $s_1, s_2 \in (0, t_1]$, and relation between s_1,s_2 is $s_1 < s_2$.

$$\begin{aligned}
\|(Nx)(s_2) - (Nx)(s_1)\| &= \|E_\alpha H t^\alpha \varphi(0) + \int_0^{s_2} (s_2 - s)^{\alpha-1} E_{\alpha,\alpha} H (s_2 - s)^\alpha [Pu(s) + f(s, x_s)]ds \\
&\quad -E_\alpha H t^\alpha \varphi(0) + \int_0^{s_1} (s_1 - s)^{\alpha-1} E_{\alpha,\alpha} H (s_1 - s)^\alpha [Pu(s) + f(s, x_s)]ds\| \\
&= \int_0^{s_2} \|(s_2 - s)^{\alpha-1} E_{\alpha,\alpha} H (s_2 - s)^\alpha [Pu(s) + f(s, x_s)]\|ds \\
&\quad - \int_0^{s_1} \|(s_1 - s)^{\alpha-1} E_{\alpha,\alpha} H (s_1 - s)^\alpha [Pu(s) + f(s, x_s)]\|ds \\
&\leq \int_0^{s_2} \|(s_2 - s)^{\alpha-1} E_{\alpha,\alpha} H (s_2 - s)^\alpha\| \|Pu(s) + f(s, x_s)\|ds \\
&\quad - \int_0^{s_1} \|(s_1 - s)^{\alpha-1} E_{\alpha,\alpha} H (s_1 - s)^\alpha\| \|Pu(s) + f(s, x_s)\|ds \\
&\leq \{\int_0^{s_2} \|(s_2 - s)^{\alpha-1} E_{\alpha,\alpha} H (s_2 - s)^\alpha\|ds - \int_0^{s_1} \|(s_1 - s)^{\alpha-1} E_{\alpha,\alpha} \\
&\quad H (s_1 - s)^\alpha\|ds\} \times [Q^\star \tilde{\kappa} + m(s)].
\end{aligned}$$

As $s_2 \to s_1$ then right hand side of the above inequality tends to zero then we have $\|(Nx)(s_2) - (Nx)(s_1)\| \to 0$ and therefore Nx is equicontinuous on $(0, t_1]$.
For $t \in (t_1, t_2]$ and let us consider the two arbitrary elements $s_2, s_3 \in (t_1, t_2]$, and relation between s_2,s_3 is $s_2 < s_3$.

$$\begin{aligned}
\|(Nx)(s_3) - (Nx)(s_2)\| &= \|E_\alpha H t^\alpha \varphi(0) + I_1 x(t_1^-) + \int_0^{s_3} (s_3 - s)^{\alpha-1} E_{\alpha,\alpha} H (s_3 - s)^\alpha \\
&\quad \times [Pu(s) + f(s, x_s)]ds - E_\alpha H t^\alpha \varphi(0) - I_1 x(t_1^-) \\
&\quad - \int_0^{s_2} (s_2 - s)^{\alpha-1} E_{\alpha,\alpha} H (s_2 - s)^\alpha [Pu(s) + f(s, x_s)]\|ds
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{s_3} \|(s_3-s)^{\alpha-1}E_{\alpha,\alpha}H(s_3-s)^{\alpha}[Pu(s)+f(s,x_s)]\|ds \\
&\quad - \int_0^{s_2} \|(s_2-s)^{\alpha-1}E_{\alpha,\alpha}H(s_2-s)^{\alpha}[Pu(s)+f(s,x_s)]\|ds \\
&\le \int_0^{s_3} \|(s_3-s)^{\alpha-1}E_{\alpha,\alpha}H(s_3-s)^{\alpha}\|\|Pu(s)+f(s,x_s)\|ds \\
&\quad - \int_0^{s_2} \|(s_2-s)^{\alpha-1}E_{\alpha,\alpha}H(s_2-s)^{\alpha}\|\|Pu(s)+f(s,x_s)\|ds \\
&\le \{\int_0^{s_3} \|(s_3-s)^{\alpha-1}E_{\alpha,\alpha}H(s_3-s)^{\alpha}\|ds - \int_0^{s_2} \|(s_2-s)^{\alpha-1}E_{\alpha,\alpha} \\
&\quad H(s_2-s)^{\alpha}\|ds\} \times [Q^{\star}\tilde{\kappa}+m(s)]. \qquad (4.9)
\end{aligned}$$

As $s_3 \to s_2$ the right side of the inequality (4.9) tends to zero then we have $\|(Nx)(s_3)-(Nx)(s_2)\| \to 0$ and therefore Nx is equicontinuous on $(t_1, t_2]$.
Proceeding like this, in the general form for every $t \in (t_i, t_{i+1}]$ then we consider the two arbitrary elements $s_m, s_{m+1} \in (t_i, t_{i+1}]$, and relation between s_m, s_{m+1} is $s_m < s_{m+1}$.

$$\begin{aligned}
\|(Nx)(s_{m+1})-(Nx)(s_m)\| &\le \{\int_0^{s_{m+1}} \|(s_{m+1}-s)^{\alpha-1}E_{\alpha,\alpha}H(s_{m+1}-s)^{\alpha}\|ds \\
&\quad - \int_0^{s_m} \|(s_m-s)^{\alpha-1}E_{\alpha,\alpha}H(s_m-s)^{\alpha}\|ds\}[Q^{\star}\tilde{\kappa}+m(s)]. \qquad (4.10)
\end{aligned}$$

As $s_{m+1} \to s_m$ then the right side of the inequality (4.10) tends to zero then we have $\|(Nx)(s_{m+1})-(Nx)(s_m)\| \to 0$ and therefore Nx is equicontinuous on $(t_i, t_{i+1}]$, the function N is bounded for every $t \in J$ and by using **Definition.(2.5)**, we obtain N is uniformly bounded and then by using Arzela-Ascoli theorem (**Lemma.(2.11)**) and **Remark.(2.12)**, we can say N is relatively compact on J. As a consequence of step 1, step 2 and step 3 together with the Schauder fixed point theorem we can presume N has a fixed point which is a solution of the equation.(1.1) on $J = [0, T]$. Q.E.D.

4.2 Controllability criterion

Theorem 4.3. The assumptions (A1-A4) are satisfied then the equation.(1.1) has unique solution on $J = [0, T]$, Provided that,

$$Q := \lambda + \int_0^t \|t^{\alpha-1}E_{\alpha,\alpha}H(t-s)^{\alpha}\|m(s)ds < 1. \qquad (4.11)$$

Proof. Let N be a function maps from $PC^1([-r, T])$ into $PC^1([-r, T])$ is well defined by **Theorem.(4.2)** then we can prove the function N has a unique fixed point. Let us take two arbitrary values $x', x'' \in J$ and for $t \in [-r, 0]$ then we get, $\|(Nx')(t)-(Nx'')(t)\| = \|\varphi(t)-\varphi(t)\| \equiv 0$.

For $t \in (0, t_1]$, by using the assumptions (A1-A4) and the inequality (4.11) then we have

$$\begin{aligned}
\|(Nx')(t)-(Nx'')(t)\| &= \|E_{\alpha}(Ht^{\alpha})\varphi(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^{\alpha}Pu(s)ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^{\alpha}f(s,x'_s)ds
\end{aligned}$$

$$
\begin{aligned}
& -E_\alpha(Ht^\alpha)\varphi(0) - \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha Pu(s)ds \\
& - \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha f(s,x_s'')ds\| \\
\leq \quad & \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha\|(\|f(s,x'_s)-f(s,x''_s)\|)ds \\
\leq \quad & \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha\|m(s)ds\|x'-x''\|_\infty \\
\leq \quad & q_1\|x'-x''\|_\infty
\end{aligned}
$$

where, $q_1 = \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha\|m(s)ds$.
For $t \in (t_1, t_2]$, by using the assumptions (A1-A4) and the inequality (4.11) then we have

$$
\begin{aligned}
\|(Nx')(t)-(Nx'')(t)\| \quad = \quad & \|E_\alpha(Ht^\alpha)\varphi(0) + I_1(x'(t_1^-)) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha Pu(s)ds \\
& + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha f(s,x_s')ds \\
& -E_\alpha(Ht^\alpha)\varphi(0) - I_1(x''(t_1^-)) - \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha Pu(s)ds \\
& - \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha f(s,x_s'')ds\| \\
\leq \quad & \|I_1(x'(t_1^-)) - I_1(x''(t_1^-))\| + \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha\| \\
& \times(\|f(s,x'_s)-f(s,x''_s)\|)ds \\
\leq \quad & \lambda_1\|x'-x''\|_\infty + \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha\|m(s)\|x'-x''\|_\infty ds \\
\leq \quad & \{\lambda_1 + \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha\|m(s)ds\}\|x'-x''\|_\infty \\
\leq \quad & q_2\|x'-x''\|_\infty
\end{aligned}
$$

where, $q_2 = \lambda_1 + \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha\|m(s)ds$.

Similarly, for $t \in (t_i, t_{i+1}]$, i=1,2,..,m by using the assumptions (A1-A4) and the inequality (4.11) then we have

$$
\begin{aligned}
\|(Nx')(t)-(Nx'')(t)\| \quad \leq \quad & \{\sum_{i=1}^m \lambda_i + \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha\|m(s)ds\}\|x'-x''\|_\infty \\
\leq \quad & q_i\|x'-x''\|_\infty
\end{aligned}
$$

where, $q_i = \sum_{i=1}^m \lambda_i + \int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^\alpha\|m(s)ds$.

$$\begin{aligned}\|(Nx')(t)-(Nx'')(t)\| &= max\{\int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^{\alpha}\|m(s)ds,\\ &\lambda_1+\int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^{\alpha}\|m(s)ds,\\ &\lambda_1+\lambda_2+\int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^{\alpha}\|m(s)ds,\\ &...,\sum_{i=1}^{m}\lambda_i+\int_0^t \|(t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^{\alpha}\|m(s)ds\}\|x'-x''\|_\infty\\ &\le max\{q_1,q_2,...q_i\}\|x'-x''\|_\infty\\ &= Q\|x'-x''\|_\infty \qquad (4.12)\end{aligned}$$

Therefore, from equation.(4.12) we have this inequality $\|(Nx')(t)-(Nx'')(t)\| \le Q\|x'-x''\|_\infty$, where $Q=\lambda+\int_0^t \|t^{\alpha-1}E_{\alpha,\alpha}H(t-s)^{\alpha}\|m(s)ds$. Since, $Q<1$ as a consequence of Banach contraction principle, we conclude that there exists a unique solution x(t) of the equation.(1.1) on $J=[0,T]$ such that

$$\begin{aligned}N(x(t)) &= E_\alpha(Ht^\alpha)\varphi(0)+\sum_{i=1}^{m}I_i(x(t_i^-))+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^{\alpha}f(s,x_s)ds\\ &+\kappa\kappa^{-1}[x_t-E_\alpha(Ht^\alpha)\varphi(0)-\sum_{i=1}^{m}I_i(x(t_i^-))-\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}H(t-s)^{\alpha}f(s,x_s)ds\\ &= x_t \qquad (4.13)\end{aligned}$$

Hence we have $N(x(t))=x_t$ and $N(x(0))=x_0$ at time t=0, which implies that the Impulsive fractional functional differential equation.(1.1) is controllable on $J=[0,T]$ by using **Definition.(2.9)**.

Q.E.D.

5 Numerical examples

Example 5.1. Let us consider the following fractional functional differential equation

$$\begin{aligned}{}^{C}D^{\frac{1}{2}}x(t) &= Hx(t)+Pu(t)+\frac{1}{3}(sinx(t-\frac{\pi}{2})), t\in[0,\frac{3\pi}{2}], t\neq\frac{i\pi}{2} \qquad (5.1)\\ \Delta x(\frac{\pi i}{2}) &= \frac{1}{3}, i=1,2\\ x(t) &= sin(t), t\in[-\frac{\pi}{2},0]\end{aligned}$$

Let us consider matrix H=$\begin{bmatrix} sin(\frac{\pi}{3}) & cos(\frac{\pi}{2})\\ cos(2\pi) & cos(\frac{\pi}{4})\end{bmatrix}$, the value of $\varphi(0)=0$, $M^\star=\frac{1}{3}$, $\alpha=\frac{1}{2}$, P=$\begin{bmatrix}0 & 1\end{bmatrix}$, and then $\|P\|=1$, The function is defined as

$$\|f(t,x_t)-f(t,y_t)\| \le 1/3\|sin(x(t-\pi/2))-sin(y(t-\pi/2))\|$$

and Let us take the value of $m(t) = \frac{1}{3}$, $m(t) \in \mathbb{R}$ and $\rho = 45$

$$\begin{aligned} E_{\alpha,\alpha}(H(t-s)^{\alpha}) &= E_{\frac{1}{2},\frac{1}{2}} H(\frac{3\pi}{2}-s)^{\frac{1}{2}} \\ &= \sum_{k=0}^{2} \frac{H^k(\frac{3\pi}{2}-s)^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+\frac{1}{2})} \end{aligned}$$

$$= \frac{1}{\sqrt{\pi}} I + \begin{bmatrix} sin(\frac{\pi}{3})(\frac{3\pi}{2}-s)^{\frac{1}{2}} & cos(\frac{\pi}{2})(\frac{3\pi}{2}-s)^{\frac{1}{2}} \\ cos(2\pi)(\frac{3\pi}{2}-s)^{\frac{1}{2}} & cos(\frac{\pi}{4})(\frac{3\pi}{2}-s)^{\frac{1}{2}} \end{bmatrix}$$

$$+ \begin{bmatrix} sin^2(\frac{\pi}{3})(\frac{3\pi}{2}-s)\frac{2}{\sqrt{(\pi)}} + A & sin(\frac{\pi}{3})cos(\frac{\pi}{2})(\frac{3\pi}{2}-s)\frac{2}{\sqrt{(\pi)}} + B \\ cos(2\pi)sin(\frac{\pi}{3})(\frac{3\pi}{2}-s)\frac{2}{\sqrt{(\pi)}} + E & cos(2\pi)cos(\frac{\pi}{2})(\frac{3\pi}{2}-s)\frac{2}{\sqrt{(\pi)}} + F \end{bmatrix}$$

$$\begin{aligned} where,\ A &= cos(\frac{\pi}{2})cos(2\pi)\frac{2}{\sqrt{(\pi)}}(\frac{3\pi}{2}-s) \\ B &= cos(\frac{\pi}{2})cos(\frac{\pi}{4})\frac{2}{\sqrt{(\pi)}}(\frac{3\pi}{2}-s) \\ E &= cos(\frac{\pi}{4})cos(2\pi)\frac{2}{\sqrt{(\pi)}}(\frac{3\pi}{2}-s) \\ F &= cos^2(\frac{\pi}{4})\frac{2}{\sqrt{(\pi)}}(\frac{3\pi}{2}-s) \end{aligned}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{(\pi)}} + 0.86(\frac{3\pi}{2}-s)^{\frac{1}{2}} + 0.845(\frac{3\pi}{2}-s) & 0 \\ (\frac{3\pi}{2}-s)^{\frac{1}{2}} + 1.773(\frac{3\pi}{2}-s) & \frac{1}{\sqrt{(\pi)}} + 0.71(\frac{3\pi}{2}-s)^{\frac{1}{2}} + 0.563(\frac{3\pi}{2}-s) \end{bmatrix}$$

By using inequalities (4.1) and (4.4), we defined as below

$$\begin{aligned} &= \frac{1}{3} + \| \begin{bmatrix} -4.160 & 0 \\ -7.379 & -2.94 \end{bmatrix} \| \times \{1 + \frac{1}{3} + \frac{1}{3} \| \begin{bmatrix} -4.160 & 0 \\ -7.379 & -2.94 \end{bmatrix} \| + \frac{1}{3}\} \leq \rho \\ &\approx 0.9438 \leq 1 \end{aligned}$$

From **Theorem.(4.2)** the fractional functional differential equation with impulses equation.(5.1) has exists a solution on $[0, \frac{3\pi}{2}]$.

Example 5.2. Let us consider the following fractional functional differential equation

$$\begin{aligned} {}^{C}D^{\frac{1}{2}}x(t) &= Hx(t) + Pu(t) + \frac{1}{15}(sinx(t-\frac{\pi}{2})), t \in [0, \frac{3\pi}{2}], t \neq \frac{i\pi}{2} \\ \Delta x(\frac{\pi i}{2}) &= \frac{1}{15}, i = 1, 2 \\ x(t) &= sin(t), t \in [-\frac{\pi}{2}, 0] \end{aligned} \tag{5.2}$$

Let us consider matrix H=$\begin{bmatrix} sin(\frac{\pi}{3}) & cos(\frac{\pi}{2}) \\ cos(2\pi) & cos(\frac{\pi}{4}) \end{bmatrix}$ and $\|f(t,x_t) - f(t,y_t)\| \leq 1/15\|x_t - y_t\|$ and where, $m(t) = \frac{1}{15}$, $m(t) \in \mathbb{R}$ and $\alpha = \frac{1}{2}$
Let us take $\varphi(0) = 0$ and P=$\begin{bmatrix} 0 & 1 \end{bmatrix}$, then $\|P\| = 1$

$$\begin{aligned} E_{\alpha,\alpha}(H(t-s)^{\alpha}) &= E_{\frac{1}{2},\frac{1}{2}} H(\frac{3\pi}{2} - s)^{\frac{1}{2}} \\ &= \sum_{k=0}^{2} \frac{H^k(\frac{3\pi}{2} - s)^{\frac{k}{2}}}{\Gamma(\frac{k}{2} + \frac{1}{2})} \\ &= \frac{1}{\sqrt{\pi}} I + \begin{bmatrix} sin(\frac{\pi}{3})(\frac{3\pi}{2} - s)^{\frac{1}{2}} & cos(\frac{\pi}{2})(\frac{3\pi}{2} - s)^{\frac{1}{2}} \\ cos(2\pi)(\frac{3\pi}{2} - s)^{\frac{1}{2}} & cos(\frac{\pi}{4})(\frac{3\pi}{2} - s)^{\frac{1}{2}} \end{bmatrix} \end{aligned}$$

$$+ \begin{bmatrix} sin^2(\frac{\pi}{3})(\frac{3\pi}{2} - s)\frac{2}{\sqrt{(\pi)}} + A & sin(\frac{\pi}{3})cos(\frac{\pi}{2})(\frac{3\pi}{2} - s)\frac{2}{\sqrt{(\pi)}} + B \\ cos(2\pi)sin(\frac{\pi}{3})(\frac{3\pi}{2} - s)\frac{2}{\sqrt{(\pi)}} + E & cos(2\pi)cos(\frac{\pi}{2})(\frac{3\pi}{2} - s)\frac{2}{\sqrt{(\pi)}} + F \end{bmatrix}$$

$$\begin{aligned} where,\ A &= cos(\frac{\pi}{2})cos(2\pi)\frac{2}{\sqrt{(\pi)}}(\frac{3\pi}{2} - s) \\ B &= cos(\frac{\pi}{2})cos(\frac{\pi}{4})\frac{2}{\sqrt{(\pi)}}(\frac{3\pi}{2} - s) \\ E &= cos(\frac{\pi}{4})cos(2\pi)\frac{2}{\sqrt{(\pi)}}(\frac{3\pi}{2} - s) \\ F &= cos^2(\frac{\pi}{4})\frac{2}{\sqrt{(\pi)}}(\frac{3\pi}{2} - s) \end{aligned}$$

By using the inequality (4.11) and we have,

$$\begin{aligned} &= \frac{1}{15} + \frac{1}{15}\| \begin{bmatrix} -4.160 & 0 \\ -7.379 & -2.94 \end{bmatrix} \| \\ &\approx 0.6795 < 1 \end{aligned}$$

From our all assumptions (A1-A4) and by using **Theorem.(4.3)** then the fractional functional differential equation.(5.2) has a unique solution such that $x(t) = \frac{1}{15} + 0.657 + x_t - \frac{1}{15} - 0.657 = x_t$ (by equation.(4.13)) on $t \in [0, \frac{3\pi}{2}]$) and by using **Definition.(2.9)**, we can say the equation.(5.2) is controllability on $[0, \frac{3\pi}{2}]$.

6 Graphical representation

During the time interval $[0, \frac{3\pi}{2}]$, the controlled system of a given dynamical system is graphically interpreted as below. **Figures.(1)** and **(2)** represents the system with and without control function steers from initial state $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ to final state $\begin{bmatrix} 8 \\ 3 \end{bmatrix}$ respectively. Similarly, **Figures.(3)** and **(4)** represents the system with and without control function from different initial state $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to different final state $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ respectively.

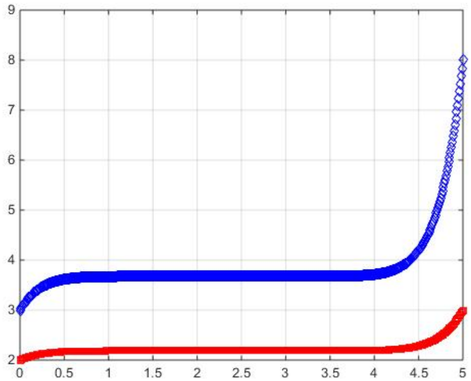

Figure 1: A system steers from initial state $\begin{bmatrix}3\\2\end{bmatrix}$ to final state $\begin{bmatrix}8\\3\end{bmatrix}$ on the interval $[0, \frac{3\pi}{2}]$.

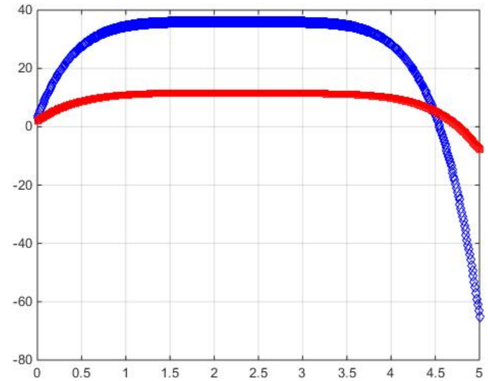

Figure 2: A system without control function on the interval $[0, \frac{3\pi}{2}]$ from initial state $\begin{bmatrix}3\\2\end{bmatrix}$ to final state $\begin{bmatrix}8\\3\end{bmatrix}$.

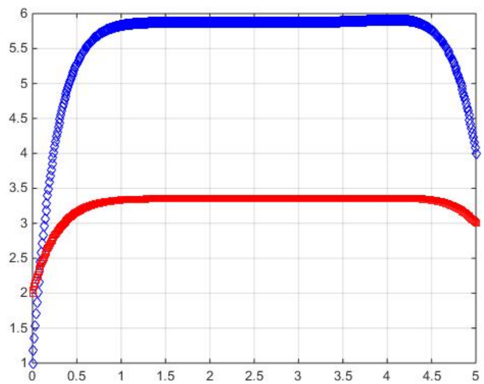

Figure 3: A system steers from initial state $\begin{bmatrix}1\\2\end{bmatrix}$ to final state $\begin{bmatrix}4\\3\end{bmatrix}$ on the interval $[0, \frac{3\pi}{2}]$.

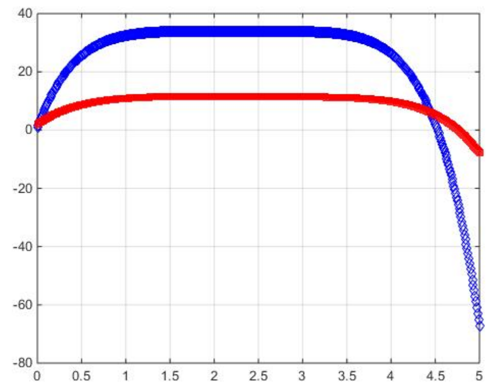

Figure 4: A system without control function on the interval $[0, \frac{3\pi}{2}]$ from initial state $\begin{bmatrix}1\\2\end{bmatrix}$ to final state $\begin{bmatrix}4\\3\end{bmatrix}$.

7 Conclusion

This manuscript effectuates the idea of controllability criteria for the impulsive fractional functional differential equation. We proved the existence, uniqueness results are attained by Schauder fixed point theorem and Banach Contraction principle respectively and also defined the linear operator for suitable control function. Then extract the controllability of fractional functional differential equation, also executed the two numerical computations with graphical representations and it is use to illustrate the effectiveness and applicability of the results obtained.

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